M150. *Proposed by Arkady Alt, San Jose, CA, USA.*

Let two complex numbers z_1 and z_2 satisfy the conditions

$$\begin{aligned} z_1 + z_2 &= -(i+1)\,, \\ z_1 \cdot z_2 &= -i\,. \end{aligned}$$

Without calculating z_1 and z_2, find $z_1 \cdot \overline{z_2}$.

Solution by the proposer.

Note that $z_1 \cdot \overline{z_2} = \frac{z_1}{z_2} \cdot |z_2|^2$. From $(z_1 + z_2)^2 = 2i = -2z_1 \cdot z_2$, we immediately obtain $z_1^2 + 4z_1z_2 + z_2^2 = 0$, or equivalently,

$$\left(\frac{z_1}{z_2}\right)^2 + 4\left(\frac{z_1}{z_2}\right) + 1 \;=\; 0\,.$$

Thus, $\frac{z_1}{z_2}$ is real and negative. Therefore, $z_1 \cdot \overline{z_2}$ is also real and negative. Combining this with $|z_1 \cdot \overline{z_2}| = |z_1 \cdot z_2| = 1$, we see that $z_1 \cdot \overline{z_2} = -1$.